TWMS J. Pure Appl. Math. V.9, N.2, 2018, pp.223-230

ON PRIMENESS OF NON-DETERMINISTIC AUTOMATA ASSOCIATED WITH INPUT SEMIGROUP S

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Abstract. We have introduced the notions of prime, semiprime, irreducible, and maximal of non-deterministic automata (NDA) and charaterize the monoid in term of these NDA's.

Keywords: semigroups, non-deterministic automata, prime and semiprime NDA's, irreducible NDA's

AMS Subject Classification: 20M30, 20M35.

1. Introduction and preliminaries

One of the very competent conception in many branches of Mathematics as well as in Computer Science is the action of a semigroup or a monoid on a set. In 1922, Suschkewitsch in his dissertation "The Theory of Action as Generalized Group Theory" introduced the notion of semigroup action [12]. A representation of semigroup S by transformation of a set defines an S-act just as representation of a ring R by endomorphisms of an Abelian group defines an R-module. An automaton without outputs can be considered as S-act and S-acts can be considered as automaton without outputs. Acts over semigroups appeared and were used in a variety of applications like algebraic automata theory, mathematical linguistics etc. The principal notions of monoid actions can be found in [6].

The theory of hyper structure was initiated by F. Marty in 1934 at the 8th congress of Scandinavian mathematics. F. Marty later investigated the structure of hyper groups and then applied them to study the structure of groups [8]. Different hyper structures are extensively studied from the theoretical perspective such as in fuzzy set theory, rough set theory, optimization theory, cryptography, codes, analysis of computer programs, automata, formal language theory, combinatorics, artificial intelligence, probability, graphs and hyper graphs, geometry, lattices and binary relations ([2]-[14] and [15]). A contemporary book [1], contains an affluence of applications. In 2011, Sen et al. introduced the notion of hyperaction of a semigroup on a non-empty set and proved that a non-deterministic automaton without outputs can be considered as S-hyper set and vice versa [9]. Consequently, throughout this paper, we will speak of non-deterministic automaton without outputs instead of S-hyper set. In 2017, Shabir et al. generalized the concept of hyper S-acts by defining the hyperaction of hyper monoid on a nonempty set and named it as GHS-acts and discussed their primeness [10]. In this paper, the concept of primeness has been studied for hyper S-acts.

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Manuscript received May 2017.

The principal notions of non-deterministic automaton without outputs (hyper S-acts) can be found in [9] and [11]. Now, we recall some basic definitions and results from the theory of non-deterministic automata which will be required in the later section.

Definition 1.1. *A non-deterministic automata or S-hyper set, $\mathcal{X} = (X, S, \eta)$, is a triplet where X is a non-empty set, S is a monoid with identity element e and η is a function from $X \times S$ into $P^*(X)$ defined by $\eta(x, s) = x * s$ for all $x \in X$ and $s \in S$. Also, we shall assume the useful properties: (i) $(x * s) * t = x * (st)$ and (ii) $x = x * e$ for all $x \in X$ and $s, t, e \in S$.*

Since the main interest is in the structure of the X and the input monoid, so outputs are not considered here. An NDA $\mathcal{X}$ means a triple (X, S, η) and X does not mean an NDA. But the attribute 'NDA' will be sometimes used for X.

Definition 1.2. *Let $\mathcal{X} = (X, S, \eta)$ be an NDA. The $\mathcal{Y} = (Y, S, \eta_{\mathcal{Y}})$ is a non-deterministic sub automata of $\mathcal{X}$ if $Y \subseteq X$ and $y * s \subseteq Y$ for all $y \in Y$ and $s \in S$.*

Definition 1.3. *Let $\mathcal{X} = (X, S, \eta)$ be an NDA. An element $\theta \in X$ is called a* ***fixed element*** *of $(X_S, *)$ if it satisfies $\theta * s = \theta$ for all $s \in S$.*

Note that an NDA may have several fixed elements, unique fixed element or it may also have no fixed element. Let $\mathcal{D}$ denote the set of all fixed elements of NDA $\mathcal{X}$. Then $\mathcal{X}$ is called **pure centered** if S is a monoid with two-sided zero element 0 and $|\mathcal{D}| = 1$, where $|\cdot|$ denotes the cardinality.

An equivalence relation σ on an NDA $\mathcal{X} = (X, S, \eta)$ is called a **congruence** if σ satisfies the following compatibility property:

CP: $x\sigma y \Longrightarrow \dfrac{x * s}{\sigma} = \dfrac{y * s}{\sigma}$ for every $x, y \in X$ and $s \in S$, that is,
for every $x_1 \in x * s$ there exists $y_1 \in y * s$ such that $x_1 \sigma y_1$ and for every
$y_2 \in y * s$ there exists $x_2 \in x * s$ such that $x_2 \sigma y_2$.

If $\mathcal{Y} = (Y, S, \eta_{\mathcal{Y}})$ is a non-deterministic subautomata of $\mathcal{X}$, then $\mathcal{Y}$ determines a congruence σ on $\mathcal{X}$ as follows:

$$\text{for } x, y \in X,\ x\sigma y \text{ if and only if either } x = y \text{ or } x, y \in Y.$$

We write $\mathcal{X}/\mathcal{Y}$ and call it **Rees factor GHS-act** of $\mathcal{X}$ by $\mathcal{Y}$.

2. Main results

Throughout this section, unless otherwise stated, S is a monoid with two sided zero 0 and all right NDA's are pure centered. We begin with the following proposition.

Proposition 2.1. *If $\mathcal{Y} = (Y, S, \eta_{\mathcal{Y}})$ is a non-deterministic subautomata of an NDA $\mathcal{X} = (X, S, \eta)$, then the set $\{s \in S : X * s \subseteq Y\}$ is an ideal of S.*

Proof. Obvious. □

The following corollary is an immediate consequence of Proposition 2.1.

Corollary 2.1. *For an NDA $\mathcal{X} = (X, S, \eta)$, the set $\{s \in S : X * s = (\theta)\}$ is an ideal of S.*

Let $\mathcal{X} = (X, S, \eta)$ be an NDA. Then the ideal $\mathcal{H}_\theta = \mathcal{H} = \{s \in S : X * s = (\theta)\}$ is called the annihilator of $\mathcal{X}$ in S. An NDA $\mathcal{X} = (X, S, \eta)$ is **faithful** if $\mathcal{H} = \{0\}$.

If $\mathcal{Y} = (Y, S, \eta_{\mathcal{Y}})$ is a non-deterministic subautomata of $\mathcal{X} = (X, S, \eta)$, then by Proposition 2.1 $\{s \in S : X * s \subseteq Y\}$ is an ideal of S, called the associated ideal. The associated ideal will be denoted by $\mathcal{H}_{\mathcal{Y}}$.

Definition 2.1. *A non-deterministic subautomata $\mathcal{Y} = (Y, S, \eta_{\mathcal{Y}})$ of $\mathcal{X} = (X, S, \eta)$ is called prime if for any $\nu \in X$ and $t \in S$, the inclusion $(\nu * S) * t \subseteq Y$ implies either $\nu \in Y$ or $t \in \mathcal{H}_{\mathcal{Y}}$. If for any $\nu \in X$ and $t \in S$, the inclusion $(\nu * t) * (S \circ t) \subseteq Y$ implies $\nu * t \subseteq Y$, then $\mathcal{Y} = (Y, S, \eta_{\mathcal{Y}})$ is called a semiprime non-deterministic subautomata of $\mathcal{X} = (X, S, \eta)$.*

An NDA $\mathcal{X} = (X, S, \eta)$ is itself called prime if $\{\theta\}$ is prime. Similarly, $\mathcal{X} = (X, S, \eta)$ is itself called semiprime if $\{\theta\}$ is semiprime.

Proposition 2.2. *If $\mathcal{Y} = (Y, S, \eta_{\mathcal{Y}})$ is a prime non-deterministic subautomata of $\mathcal{X} = (X, S, \eta)$, then $\mathcal{Y}$ is semiprime.*

Proof. Suppose that $\mathcal{Y}$ is a prime. For $x \in X$, and $t \in S$, consider the inclusion $(x * t) * (S \circ t) \subseteq Y$. Since $\mathcal{Y}$ is a prime non-deterministic subautomata of $\mathcal{X}$, it follows that either $x * t \subseteq Y$ or $t \in \mathcal{H}_Y$. Suppose $x * t \nsubseteq Y$. Then $x * t \subseteq X * t$ and $X * t \nsubseteq Y$. Hence $t \notin \mathcal{H}_Y$ which contradicts the assumption that $\mathcal{Y}$ is prime. So $x * t \subseteq Y$ and hence $\mathcal{Y}$ is semiprime. □

Proposition 2.3. *Every nonzero non-deterministic subautomata $\mathcal{Y} = (Y, S, \eta_{\mathcal{Y}})$ of a prime non-deterministic automata $\mathcal{X} = (X, S, \eta)$ is a prime.*

Proof. Suppose for $t \in S$ and $\nu \in X$, we have $(\nu * S) * t = \{\theta\}$. If $\nu \neq \theta$, then since $\mathcal{X}$ is a prime (that is, $\{\theta\}$ is a prime non-deterministic subautomata of $\mathcal{X}$), it follows that

$$t \in \mathcal{H}_\theta = \{s \in S : X * s = \{\theta\}\} \subseteq \{s \in S : Y * s = \{\theta\}\}.$$

Hence $\mathcal{Y}$ is prime. □

The next result shows a close connection between prime ideal and prime non-deterministic automata.

Theorem 2.1. *Let $\mathcal{Y} = (Y, S, \eta_{\mathcal{Y}})$ be a non-deterministic subautomata of $\mathcal{X} = (X, S, \eta)$. If $\mathcal{Y}$ is prime, then the associated ideal $\mathcal{H}_B$ of $\mathcal{Y}$ is a prime ideal of S.*

Proof. Consider the inclusion $tSt' \subseteq \mathcal{H}_{\mathcal{Y}}$ for $t, t' \in S$. Assume $t \notin \mathcal{H}_Y$. Then $X * t \nsubseteq B$. Hence there exists $x \in X$ such that $x * t \nsubseteq B$. Since $tSt' \subseteq \mathcal{H}_Y$; $X * (tSt') \subseteq Y$ which implies $((x * t) * S) * t' \subseteq Y$. Since $\mathcal{Y}$ is a prime non-deterministic subautomata of $\mathcal{X}$ and $x * t \nsubseteq Y$, we get $t' \in \mathcal{H}_{\mathcal{Y}}$, that is, $\mathcal{H}_{\mathcal{Y}}$ is a prime ideal. □

Proposition 2.4. *Let $\mathcal{X} = (X, S, \eta)$ be an NDA. Then for any non-deterministic subautomata $\mathcal{Y} = (Y, S, \eta_{\mathcal{Y}})$ of $\mathcal{X}$ and the corresponding associated ideal $\mathcal{H}_{\mathcal{Y}}$, the following assertions are equivalent:*

(i) $\mathcal{Y}$ is a prime,
(ii) for all non-deterministic subautomata $\mathcal{Z} = (Z, S, \eta_{\mathcal{Z}})$ of $\mathcal{X}$ and right ideal I of S, $Z * I \subseteq Y$ implies either $\mathcal{Z} \subseteq \mathcal{Y}$ or $I \subseteq \mathcal{H}_{\mathcal{Y}}$.

Proof. (i)$\Longrightarrow$(ii) Consider $Z * I \subseteq Y$ for non-deterministic subautomata $\mathcal{Z} = (Z, S, \eta_{\mathcal{Z}})$ of $\mathcal{X}$ and ideal I of S. This implies $z * t \subseteq Y$ for $z \in Z$ and $t \in I$. So

$$\begin{aligned} z * t &= z * (et) = (z * e) * t \\ &\subseteq (z * S) * t \subseteq Z * I \subseteq Y. \end{aligned}$$

Since $\mathcal{Y}$ is prime, $(z * S) * t \subseteq Y$ implies either $z \in Y$ or $t \in \mathcal{H}_{\mathcal{Y}}$, that is,either $\mathcal{Z} \subseteq \mathcal{Y}$ or $I \subseteq \mathcal{H}_{\mathcal{Y}}$.

(ii)$\Longrightarrow$(i) Consider the inclusion $(x * S) * t \subseteq Y$ for $x \in X \backslash Y$ and $t \in S$. Now $((x * S) * t) * S \subseteq Y * S \subseteq Y$ implies $(x * S) * (tS) \subseteq Y$. Set $Z = x * S$ and $I = tS$. Since $Z * I \subseteq Y$ and $Z \nsubseteq Y$, from (ii) we have, $t \in I \subseteq \mathcal{H}_{\mathcal{Y}}$, that is, $\mathcal{Y}$ is prime. □

Proposition 2.5. *Let $\mathcal{Z} = (Z, S, \eta_{\mathcal{Z}})$ be a non-deterministic subautomata of $\mathcal{X} = (X, S, \eta)$. Then $\mathcal{Z}$ is prime if and only if associated ideal $\mathcal{H}_{\mathcal{Z}}$ of $\mathcal{Z}$ in $\mathcal{Y}$ is same as associated ideal $\mathcal{H}'_{\mathcal{Z}}$ of $\mathcal{Z}$ in $\mathcal{X}$ for all non-deterministic subautomata $\mathcal{Y} = (Y, S, \eta_{\mathcal{Y}})$ with $\mathcal{Z} \subseteq \mathcal{Y} \subseteq \mathcal{X}$.*

Proof. If $\mathcal{Z}$ is prime and $\mathcal{Y} = (Y, S, \eta_{\mathcal{Y}})$ be any non-deterministic subautomata of $\mathcal{X}$ with $\mathcal{Z} \subseteq \mathcal{Y} \subseteq \mathcal{X}$, then $\mathcal{H}'_{\mathcal{Z}} \subseteq \mathcal{H}_{\mathcal{Z}}$. Let $s \in S$ such that $Y * s \subseteq Z$. For some $x \in Y \backslash Z$, we have

$$\begin{aligned} x * s &= x * (es) = (x * e) * s \subseteq (x * S) * s \\ &\Longrightarrow (x * S) * s \subseteq (Y * S) * s \subseteq Y * s \subseteq Z. \end{aligned}$$

Since $\mathcal{Z}$ is prime with $x \in Y \backslash Z$, $(x * S) * s \subseteq Z$ implies $s \in \mathcal{H}_{\mathcal{Z}}$, that is, $\mathcal{H}_{\mathcal{Z}} \subseteq \mathcal{H}'_{\mathcal{Z}}$.

Conversely, suppose that the associated ideal of $\mathcal{Z}$ in $\mathcal{Y}$ is same as associated ideal of $\mathcal{Z}$ in $\mathcal{X}$ for any non-deterministic subautomata $\mathcal{Y}$ with $\mathcal{Z} \subseteq \mathcal{Y} \subseteq \mathcal{X}$. Consider the inclusion $(x * S) * t \subseteq Z$ for $x \in X \backslash Z$ and $t \in S$. Therefore,

$$x * t = (x * e) * t \subseteq (x * S) * t \subseteq Z.$$

Set $Y = Z \cup \{x\}$. Then, clearly $\mathcal{Y} = (Y, S, \eta_{\mathcal{Y}})$ is a non-deterministic subautomata of $\mathcal{X}$ and

$$Y * t = (Z \cup \{x\}) * t \subseteq Z$$

which implies $t \in \mathcal{H}_{\mathcal{Z}}$. Therefore, by assumption $t \in \mathcal{H}'_{\mathcal{Z}}$, that is, $\mathcal{Z}$ is prime. □

The following corollary is an immediate consequence of above proposition.

Corollary 2.2. *An NDA is prime if and only if every nonzero subautomata has the same associated ideal.*

Corollary 2.3. *A NDA $\mathcal{X} = (X, S, \eta)$ is prime if and only if for every subautomata $\mathcal{Y} = (Y, S, \eta_{\mathcal{Y}})$ of $\mathcal{X}$ the annihilators of $\mathcal{X}$ and $\mathcal{X}$ in S are identical.*

Proof. Let $\mathcal{Y} = (Y, S, \eta_{\mathcal{Y}})$ be a proper subautomata of $\mathcal{X}$ and $\mathcal{H}_\theta$, $\mathcal{H}'_\theta$ be the annihilators of $\mathcal{X}$ and $\mathcal{Y}$, respectively. Then $\mathcal{H}_\theta \subseteq \mathcal{H}'_\theta$. Now, take $s \in S$ such that $Y * s = \{\theta\}$. As $(Y * S) * s \subseteq Y * s = \{\theta\}$ and $\mathcal{X}$ is prime GHS-act, so $s \in H_\theta$, that is , $\mathcal{H}'_\theta \subseteq \mathcal{H}_\theta$.

Converse is obvious. □

The next theorem gives the condition for the existence of a faithful prime non-deterministic automata.

Theorem 2.2. *A monoid S is prime if and only if there exists a faithful prime NDA.*

Proof. If S is a prime monoid, then $\mathcal{H}_0 = \{0\}$ is prime as subautomata of $(S, S, \cdot)$. Therefore, S is a faithful prime non-deterministic automata.

Conversely, let $\mathcal{X} = (X, S, \eta)$ be a faithful prime non-deterministic automata. We show that monoid S is prime, that is, $\{0\}$ is prime ideal. Suppose that $tSt' = \{0\}$ for $t, t' \in S$. If $t \neq 0$, then $X * (t \circ S) \neq \{\theta\}$. For if $X * (tS) = \{\theta\}$, then $tS \subseteq \{s \in S : X * s = \{\theta\}\} = \{0\}$. Thus $t = 0$, a contradiction to the assumption. So there exists $x \in X$ such that $x * (tS) = (x * t) * S \neq \{\theta\}$ which implies $x * t \neq \theta$. But $tSt' = (0)$. Hence $x * (t \circ S \circ t') = \{\theta\}$. Since $\mathcal{X}$ is prime and $((x * t) * S) * t' = \{\theta\}$ with $x * t \neq \theta$ which implies that $t' \in \mathcal{H}_\theta = \{s \in S : X * s = \{\theta\}\} = \{0\}$. Hence $\{0\}$ is a prime ideal of S. □

The next theorem gives the condition under which an NDA is prime.

Theorem 2.3. *Let $\mathcal{X} = (X, S, \eta)$ be an NDA. Then any non-deterministic subautomata $\mathcal{Y}$ of $\mathcal{X}$ is prime if and only if $\mathcal{X}/\mathcal{Y}$is a prime.*

Proof. Let $\mathcal{Y} = (Y, S, \eta_{\mathcal{Y}})$ be a prime subautomata of $\mathcal{X}$. Suppose that $\left(\frac{x}{Y} *_{X/Y} S\right) *_{X/Y} t = \left\{\theta_{X/Y}\right\} = \{Y\}$ with $\frac{x}{Y} \neq Y$, that is, $\frac{x}{Y} = \{x\} \nsubseteq Y$. Hence, $\frac{(x*S)*t}{Y} = Y$ implies $(x * S) * t \subseteq Y$ with $x \notin Y$ and the fact that $\mathcal{Y}$ is prime subautomata of $\mathcal{X}$ implies $X * t \subseteq Y$. Therefore, $\mathcal{X}/\mathcal{Y} = (X/Y, S, *_{X/Y})$ is a prime.

Conversely, if $(x * S) * t \subseteq Y$ and $x \notin Y$ then $\frac{x}{Y} = \{x\}$ and $\frac{(x*S)*t}{Y} = Y$. Since $\mathcal{X}/\mathcal{Y}$ is prime, we have $\frac{x}{Y} = Y$ (contrast to the assumption) or $X/Y *_{X/Y} t = Y$. Therefore, $X * t \subseteq Y$ which implies $\mathcal{Y}$ is prime. $\square$

Proposition 2.6. *For any ideal I of a monoid S, the following assertions are equivalent:*

(i) I is prime ideal,
(ii) there exists a prime NDA $\mathcal{X} = (X, S, \eta)$ with $I = \mathcal{H}_\theta = \{s \in S : X * s = \{\theta\}\}$.

Proof. Suppose I is a prime ideal of S. Then the Rees factor monoid S/I is a prime and thus by Proposition 2.2, there exists a faithful prime nondeterministic automata such that $\left\{\frac{s}{I} \in S/I : X * \frac{s}{I} = \frac{\theta}{I}\right\}$. From this it follows that $I = \mathcal{H}_\theta = \{s \in S : X * s = (\theta_X)\}$.

Conversely, suppose $\mathcal{X} = (X, S, \eta)$ is a prime non-deterministic automata with $I = \mathcal{H}_\theta$. Thus $\mathcal{X}$ is prime over the Rees factor monoid S/I which is faithful. Hence by Proposition 2.2, S/I is prime monoid and therefore, I is a prime ideal of S. $\square$

Proposition 2.7. *Let $\mathcal{X} = (X, S, \eta)$ be a finitely generated right NDA over a monoid S. Then every proper non-deterministic subautomata of $\mathcal{X}$ is contained in a maximal non-deterministic subautomata $\mathcal{X}$.*

Proof. Let $\mathcal{Y} = (Y, S, \eta_{\mathcal{Y}})$ be a proper non-deterministic subautomata of finitely generated NDA $\mathcal{X}$. Then there is finite set $\{x_1, x_2, \cdots, x_n\}$ of elements of X such that

$$X = Y \cup (x_1 * S) \cup (x_2 * S) \cup \cdots \cup (x_n * S),$$

where each $x_i * S$ $(i = 1, 2, 3, ..., n)$ is a non-deterministic subautomata of $\mathcal{X}$. There may exists such subsets from which we choose one of minimal order. Assume that $\{x_1, x_2, \cdots, x_n\}$ has minimal order. Consider the non-deterministic subautomata

$$Z = Y \cup (x_2 * S) \cup (x_3 * S) \cup \cdots \cup (x_n * S),$$

of $\mathcal{X}$ containing Y. Clearly, $\mathcal{Z} = (Z, S, \eta_{\mathcal{Z}})$ is a proper non-deterministic subautomata of $\mathcal{X}$. Otherwise, the set $\{x_2, x_3, \cdots, x_n\}$ would act for $\{x_1, x_2, \cdots, x_n\}$. Let $\mathfrak{P}$ be the collection of all non-deterministic subautomata of $\mathcal{X}$ containing Z. Clearly, $\mathfrak{P}$ is a non-empty partially ordered subset of the lattice of non-deterministic subautomata of $\mathcal{X}$ that contain Z. An non-deterministic subautomata $\mathcal{Z}'$ that contains Z is in $\mathfrak{P}$ if and only if $x_1 * S \nsubseteq Z'$. Suppose $\mathfrak{h}$ is a non-empty chain in $\mathfrak{P}$. Then $\bigcup_{\mathcal{Z}_1 \in \mathfrak{h}} \mathcal{Z}_1$ is an non-deterministic subautomata of $\mathcal{X}$ not containing x_1. Therefore, by Zorn's lemma, $\mathfrak{P}$ contains a maximal element, say $\mathcal{W}$. Since $\mathcal{W}$ is maximal in $\mathfrak{P}$, so any strictly larger non-deterministic subautomata of $\mathcal{X}$ is not in $\mathfrak{P}$ and so contains $x_1 * S$. Then any such non-deterministic subautomata must contain $W \cup (x_1 * S) \supseteq Z' \cup (x_1 * S) = X$ because $W_S \supseteq' Z'_S$. Thus $\mathcal{W}$ is maximal (proper) non-deterministic subautomata of $\mathcal{X}$ containing Y. $\square$

Proposition 2.8. *Every maximal non-deterministic subautomata of an NDA is prime.*

Proof. Let $\mathcal{Y} = (Y, S, \eta_{\mathcal{Y}})$ be a maximal non-deterministic subautomata of NDA $\mathcal{X} = (X, S, \eta)$. For $x \in X$ and $t \in S$, consider the inclusion $(x * S) * t \subseteq Y$ with $x \notin Y$. Since $\mathcal{Y}$ is maximal non-deterministic subautomata of $\mathcal{X}$ and $x \notin Y$ which implies $Y \cup (x * S) = X$. Let y be an arbitrary element of X. Then $y \in Y$ or $y \in x * S$. Thus $y = y'$ for some $y' \in Y_S$ or $y \in x * t'$ for some $t' \in S$. Then

$$y * t = y' * t \subseteq Y \quad \text{or}$$
$$y * t \subseteq \big(x * t'\big) * t = x * \big(t' \circ t\big) \subseteq x * (S \circ t) \subseteq Y.$$

Hence, $y * t \subseteq Y$ for all $y \in X$ which implies $t \in \mathcal{H}_{\mathcal{Y}}$. Therefore, $\mathcal{Y}$ is a prime non-deterministic subautomata. □

Combining Propositions 2.7 and 2.8, we obtain the following result.

Theorem 2.4. *Every proper non-deterministic subautomata of a finitely generated NDA is contained in a prime NDA.*

A non-deterministic subautomata $\mathcal{Y} = (Y, S, \eta_{\mathcal{Y}})$ of an NDA $\mathcal{X} = (X, S, \eta)$ is **irreducible** if it cannot be written as the intersection of two non-deterministic subautomatas of $\mathcal{X}$ in which it is properly contained. In other words, for any two non-deterministic subautomatas $\mathcal{Z}$ and $\mathcal{W}$ of $\mathcal{X}$,

$$Y = Z \cap W \text{ implies that either } Y = Z \text{ or } Y = W.$$

Proposition 2.9. *Every proper non-deterministic subautomata $\mathcal{Y} = (Y, S, \eta_{\mathcal{Y}})$ of an NDA $\mathcal{X} = (X, S, \eta)$ is the intersection of all the irreducible NDAs containing Y.*

Proof. For $x \in X \backslash Y$, let $\mathcal{Z}_x$ be any non-deterministic subautomata of $\mathcal{X}$ maximal with respect to $\mathcal{Y} \subseteq \mathcal{Z}_x$ but $x \notin Z_x$. Suppose $\mathcal{Z}_x = \mathcal{W} \cap \mathcal{W}'$ for GHS-subacts $\mathcal{W}$ and $\mathcal{W}'$ of $\mathcal{X}$ with $\mathcal{W} \neq \mathcal{Z}_x$ and $\mathcal{W}' \neq \mathcal{Z}_x$. The maximality of $\mathcal{Z}_x$ implies that $x \in W$ and $x \in W'$. But then $x \in W \cap W' = Z_x$ which is a contradiction. Thus $\mathcal{Z}_x$ is irreducible and $\mathcal{Y} = \cap\{\mathcal{Z}_x : x \in X \backslash Y\}$. □

A semiprime non-deterministic subautomata need not to be a prime. Next proposition gives the criteria under which a semiprime non-deterministic subautomata is prime.

Proposition 2.10. *For an irreducible non-deterministic subautomata $\mathcal{Y} = (Y, S, \eta_{\mathcal{Y}})$ of an NDA $\mathcal{X} = (X, S, \eta)$ the following assertions are equivalent:*

(i) $\mathcal{Y}$ is prime.
(ii) $\mathcal{Y}$ is semiprime.

Proof. (i)$\Longrightarrow$(ii): It follows from the Proposition 2.2.

(ii)$\Longrightarrow$(i): Assume that $\mathcal{Y}$ is a semiprime non-deterministic subautomata of $\mathcal{X}$ which is not prime. Then there exists $x \in X$ and $t \in S$ such that $x * (S \circ t) \subseteq Y$ with $x \notin Y$ and $t \notin \mathcal{H}_{\mathcal{Y}}$. Since $t \notin \mathcal{H}_{\mathcal{Y}}$, that is, $X * t \nsubseteq Y$. Hence, there exists $x' \in X$ such that $x' * t \nsubseteq Y$. Let $z \in x' * t \subseteq X$ and $((x * S) \cup Y) \cap ((z * S) \cup Y) \nsubseteq Y$. For if $((\nu * S) \cup Y) \cap ((z * S) \cup Y) \subseteq Y$, then since $\mathcal{Y}$ is irreducible, so either $(x * S) \cup Y = Y$ or $(z * S) \cup Y = Y$. This implies either $x * S \subseteq Y$ or $z * S \subseteq Y$. Hence, either $x \in x * S \subseteq Y$ or $z \in z * S \subseteq Y$, a contradiction. So there exists $s \in S$ such that $z * s \subseteq x * S$ but $z * s \nsubseteq Y$. Now

$$((z * s) * S) * (t \circ s) \subseteq (((x * t) * s) * S) * (t \circ s) \subseteq ((x * S) * t) * s \subseteq Y * s \subseteq Y$$

but

$$\big(x' * t\big) * s = x' * (t \circ s) \nsubseteq Y.$$

This implies that $\mathcal{Y}$ is not semiprime, a contradiction to our assumption. Hence $\mathcal{Y}$ is prime. □

Theorem 2.5. *Let $\mathcal{X} = (X, S, \eta)$ be an NDA over a monoid S. Then the following conditions are equivalent:*

(i) Every proper non-deterministic subautomata of $\mathcal{X}$ is semiprime.
(ii) Every proper non-deterministic subautomata of $\mathcal{X}$ is the intersection of prime NDSAs of $\mathcal{X}$.

Proof. (i)⇒(ii) Let $\mathcal{Y} = (Y, S, \eta_\mathcal{Y})$ be a proper non-deterministic subautomata of $\mathcal{X}$. Then by Proposition 2.9, $\mathcal{Y} = \cap \mathcal{Z}_x$, where each $\mathcal{Z}_x$ is proper non-deterministic subautomata of $\mathcal{X}$. Therefore, by assumption and Proposition 2.10, each $\mathcal{Z}_x$ is prime.

(ii)⇒(i) Suppose $\mathcal{Y} = \bigcap\limits_i \mathcal{Z}_i$ is an intersection of prime non-deterministic subautomatas of $\mathcal{X}$. Consider the inclusion $(x * t) * (S \circ t) \subseteq Y$ for $x \in X$ and $t \in S$. Then $(x * t) * (S \circ t) \subseteq Z_i$ for each i. By Proposition 2.9, each $\mathcal{Z}_i$ is semiprime. Hence, for each i, $x * t \subseteq Z_i$, that is, $x * t \subseteq \bigcap\limits_{i \in I} Z_i$. Therefore, $\mathcal{Y}$ is semiprime. □

Example 1. Consider the non-deterministic automata $\mathcal{X} = (X, S, \eta)$ with transition diagram and input semigroup S with the following multiplication table:

Table 1.

$*$	0	e	s	t	q
0	0	0	0	0	0
e	0	e	s	t	q
s	0	s	s	s	s
t	0	t	t	t	t
q	0	q	q	q	q

The non-deterministic subautomata of $\mathcal{X}$ are $\{\theta\}, \{\theta, x\}, \{\theta, y\}, \{\theta, x, y\}, \{\theta, x, z\}$ and X. $\{\theta, x\}$ is a subautomata of $\mathcal{X}$ which is semiprime but not prime. Indeed, Indeed, $(y * S) * t \subseteq \{\theta, x\}$ but neither $y \in \{\theta, x\}$ nor $t \in \mathcal{H}_{\{\theta, x\}} = \{0\}$. Note that $\{\theta, x\}$ is not an irreducible subautomata of $\mathcal{X}$. All other subautomatas of $\mathcal{X}$ are prime, semiprime and irreducible.

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